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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

ON MINIMUM TIME EXOATMOSPHERIC INTERCEPTION WITH FUEL CONSTRAINTS: A CLOSED LOOP SOLUTION

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ABSTRACT

The problem considered is that of minimum time interception, subject to fuel constraints, of a nonmaneuvering target by an interceptor. The motion of the interceptor is controlled by application of a thrust vector (with bounded magnitude) and by the control of the thrusting angle relative to, say, the interceptor velocity vector. This problem is solved using the minimum principle of Pontryagin[2]. It is shown that the time-optimal interception policy is (a) to use maximum thrust in the beginning of the control interval and (b) to use a constant thrust angle during the thrusting interval. The optimal thrust angle is determined analytically as a function of the position error, the velocities, and of the remaining fuel.

Accepted for the Air Force
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On Minimum Time Exoatmospheric Interception
with Fuel Constraints: A Closed Loop Solution

I THE EQUATIONS OF MOTION

The interception problem considered takes place outside the atmosphere. Only the planar version of the problem is considered. The geometry of the problem is illustrated in Fig. 1.

Consider a point target T. Let $x_T(t)$ and $y_T(t)$ denote the horizontal and vertical coordinates, respectively, of the target at time t. Let $V_{xT}(t)$ and $V_{yT}(t)$ denote the target horizontal and vertical velocities. Then, the target motion is defined by the four differential equations

$$\left.\begin{aligned} \dot{x}_T(t) &= V_{xT}(t) \\ \dot{y}_T(t) &= V_{yT}(t) \\ \dot{V}_{xT}(t) &= 0 \\ \dot{V}_{yT}(t) &= g \end{aligned}\right\} \qquad (1.1)$$

where g is the acceleration of gravity.

Next consider the interceptor I. Let $x_I(t)$ and $y_I(t)$ denote its position coordinates in the x-y plane and let $V_{xI}(t)$ and $V_{yI}(t)$ its velocity components and let m(t) denote its mass. Suppose that a thrust can be generated by the burning of fuel; let c denote the exit velocity of the propellant (assumed constant). Let $\beta(t)$ denote the rate-of-flow of fuel of the propellant. The thrust f(t) is then defined by

$$f(t) = c\beta(t) \qquad (1.2)$$

Let $\psi(t)$ denote the thrust angle with respect to the horizontal.

Using these variables, the motion of the interceptor is governed by the following set of differential equations[2]

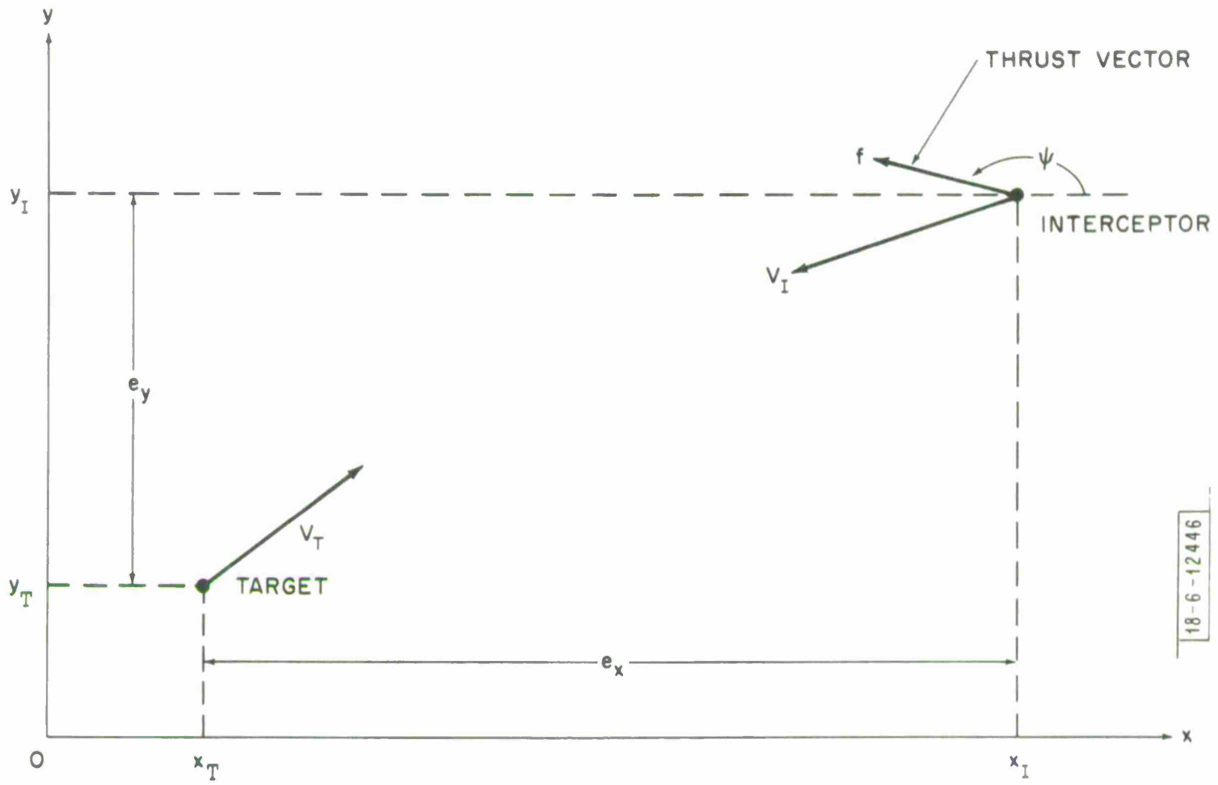

Fig. 1. The geometry of the interception problem.

$$\begin{aligned}
\dot{x}_I(t) &= V_{xI}(t) \\
\dot{y}_I(t) &= V_{yI}(t) \\
\dot{V}_{xI}(t) &= \frac{f(t)}{m(t)} \cos \psi(t) \\
\dot{V}_{yI}(t) &= \frac{f(t)}{m(t)} \sin \psi(t) - g \\
\dot{m}(t) &= \frac{-1}{c} f(t)
\end{aligned} \tag{1.3}$$

The differential equations for the interception problem can be simplified by defining the following <u>position and velocity error variables</u>

$$\begin{aligned}
e_x(t) &\triangleq x_I(t) - x_T(t) \\
e_y(t) &\triangleq y_I(t) - y_T(t) \\
v_x(t) &\triangleq V_{xI}(t) - V_{xT}(t) \\
v_y(t) &\triangleq V_{yI}(t) - V_{yT}(t)
\end{aligned} \tag{1.4}$$

From Eqs. (1.1), (1.3), and (1.4) one can verify that the error variables and the mass satisfy the differential equations

$$\begin{aligned}
\dot{e}_x(t) &= v_x(t) \\
\dot{e}_y(t) &= v_y(t) \\
\dot{v}_x(t) &= \frac{f(t)}{m(t)} \cos \psi(t) \\
\dot{v}_y(t) &= \frac{f(t)}{m(t)} \sin \psi(t) \\
\dot{m}(t) &= -\frac{f(t)}{c}
\end{aligned} \tag{1.5}$$

The five-dimensional vector

$$\begin{bmatrix} e_x(t) \\ e_y(t) \\ v_x(t) \\ v_y(t) \\ m(t) \end{bmatrix} \triangleq \underline{x}(t) \tag{1.6}$$

shall serve as the state vector for the optimization problem.

There are two constraints that will be imposed. The first constraint is that the thrust that is generated is bounded by the relation

$$0 \leq f(t) \leq F \qquad \text{for all } t \tag{1.7}$$

The second constraint is that the mass of the available propellant is limited. Thus, if m_e is the mass of the interceptor without fuel and m_o is the mass of the interceptor fully fueled, then the mass of the fuel is $m_o - m_e$.

The problem of interception in minimum time can now be posed as follows:

Problem 1.1 Given the system described by the differential equations (1.5). Suppose that at the initial time $t = 0$ the values of $e_x(0)$, $e_y(0)$, $v_x(0)$, $v_y(0)$, and $m(0) = m_o$ are known. Then, determine

(a) the thrust $f(t)$ $(0 \leq f(t) \leq F)$, and

(b) the thrust angle $\psi(t)$

so that

$$e_x(T^*) = e_y(T^*) = 0 \tag{1.8}$$

$$m(T^*) \geq m_e \tag{1.9}$$

where T^* is the minimum possible interception time.

In the following section, the minimum principle[1] will be applied to determine the necessary conditions for optimality. The necessary condition will then be used to determine the optimal thrust and the optimal thrust angle.

II THE NECESSARY CONDITIONS FOR OPTIMALITY

In this section, several relations will be developed which characterize the optimal thrust, thrust angle and the time-optimal trajectory. These relations are obtained by a straightforward application of the minimum principle to the Problem 1.1.

Let $p_1(t)$, $p_2(t)$, $p_3(t)$, $p_4(t)$, and $p_5(t)$ denote the costate variables (Lagrange multipliers) associated with the state variables $e_x(t)$, $e_y(t)$, $v_x(t)$, $v_y(t)$, and $m(t)$, respectively. Since Problem 1.1 is a time-optimal problem, the Hamiltonian function is given by

$$H = 1 + \dot{e}_x(t)p_1(t) + \dot{e}_y(t)p_2(t) + \dot{v}_x(t)p_3(t) + \dot{v}_y(t)p_4(t) + \dot{m}(t)p_5(t) \tag{2.1}$$

which upon substitution of Eq. (1.5) yields

$$H = 1 + v_x(t)p_1(t) + v_y(t)p_2(t) + \frac{f(t)}{m(t)} p_3(t) \cos \psi(t) + \frac{f(t)}{m(t)} p_4(t) \sin \psi(t) - \frac{f(t)}{c} p_5(t) \tag{2.2}$$

The costate variables satisfy the differential equations

$$\dot{p}_1(t) = -\frac{\partial H}{\partial e_x(t)} = 0 \tag{2.3a}$$

$$\dot{p}_2(t) = -\frac{\partial H}{\partial e_y(t)} = 0 \tag{2.3b}$$

$$\dot{p}_3(t) = -\frac{\partial H}{\partial v_x(t)} = -p_1(t) \tag{2.3c}$$

$$\dot{p}_4(t) = -\frac{\partial H}{\partial v_y(t)} = -p_2(t) \tag{2.3d}$$

$$\dot{p}_5(t) = -\frac{\partial H}{\partial m(t)} = \frac{f(t)}{m^2(t)} [p_3(t)\cos \psi(t) + p_4(t)\sin \psi(t)] \tag{2.3e}$$

from which one immediately deduces that

$$\left.\begin{aligned} p_1(t) &= \pi_1 \text{ (constant)} \\ p_2(t) &= \pi_2 \text{ (constant)} \\ p_3(t) &= \pi_3 - \pi_1 t \\ p_4(t) &= \pi_4 - \pi_2 t \end{aligned}\right\} \tag{2.4}$$

The posed boundary conditions at the terminal time T^* and the transversality conditions yield

$$e_x(T^*) = e_y(T^*) = 0 \tag{2.5}$$

$$p_3(T^*) = p_4(T^*) = 0 \tag{2.6}$$

$$p_5(T^*) = \begin{cases} 0 & \text{if } m(T^*) > m_e \\ ? & \text{if } m(T^*) = m_e \text{ (or } f(T^*) = 0) \end{cases} \tag{2.7}$$

Since the terminal time T^* is free and since the Hamiltonian is not an explicit function of time, then along the optimal trajectory

$$H = 0 \quad \text{for all} \quad t \in [0,T^*] \tag{2.8}$$

Let $\psi^*(t)$ and $f^*(t)$ denote the optimal thrust angle and thrust, respectively; according to the minimum principle[1] they must absolutely minimize the Hamiltonian. Noting that

$$\frac{\partial H}{\partial \psi(t)} = -\frac{f(t)}{m(t)} p_3(t) \sin \psi(t) + \frac{f(t)}{m(t)} p_4(t) \cos \psi(t) \tag{2.9}$$

$$\frac{\partial^2 H}{\partial \psi(t)^2} = -\frac{f(t)}{m(t)} p_3(t) \cos \psi(t) - \frac{f(t)}{m(t)} p_4(t) \sin \psi(t) \tag{2.10}$$

then it is necessary that

$$\left.\frac{\partial H}{\partial \psi(t)}\right|_{\psi(t)=\psi^*(t)} = 0 \tag{2.11}$$

$$\left.\frac{\partial^2 H}{\partial \psi(t)^2}\right|_{\psi(t)=\psi^*(t)} \geq 0 \tag{2.12}$$

From Eqs. (2.8) and (2.11) one deduces that

$$\tan \psi^*(t) = \frac{p_4(t)}{p_3(t)} \tag{2.13}$$

and from Eqs. (2.10) and (2.12) that

$$p_3(t) \cos \psi^*(t) + p_4(t) \sin \psi^*(t) \leq 0 \tag{2.14}$$

Equation (2.13) yields the optimal thrust-angle in terms of the costate variables. It remains to determine the relationship satisfied by the thrust $f^*(t)$. Towards this end, define the <u>switching function</u> $s(t)$ by

$$s(t) \triangleq \frac{p_3(t)}{m(t)} \cos \psi(t) + \frac{p_4(t)}{m(t)} \sin \psi(t) - \frac{p_5(t)}{c} \tag{2.15}$$

so that the Hamiltonian (2.2) takes the form

$$H = 1 + v_x(t)\, p_1(t) + v_y(t)\, p_2(t) + f(t)\, s(t) \tag{2.16}$$

The requirement that $f^*(t)$ minimizes the Hamiltonian yields the relation

$$f^*(t) = \begin{cases} F & \text{when} \quad s(t) < 0 \\ 0 & \text{when} \quad s(t) > 0 \\ ? & \text{when} \quad s(t) = 0 \end{cases}$$

Note that if $s(t) = 0$ over a finite time-interval, then Eq. (2.17c) corresponds to the singular condition.

This set of equations completes the list of relations that must hold along the optimal trajectory. In the next section, these relations will be used to deduce additional properties of the optimal thrust-angle $\psi^*(t)$ and of the optimal thrust $f^*(t)$.

III. PROPERTIES OF $\psi^*(t)$ AND $f^*(t)$

First, it will be shown that the optimal thrust angle $\psi^*(t)$ must be constant throughout the time of thrusting. This conclusion can be deduced by the following argument.

Recall that (see Eq. (2.4)) the costate variables $p_3(t)$ and $p_4(t)$ are given by

$$\left.\begin{aligned} p_3(t) &= \pi_3 - \pi_1 t \\ p_4(t) &= \pi_4 - \pi_2 t \end{aligned}\right\} \tag{3.1}$$

Use of the transversality condition (2.6), i.e.

$$p_3(T^*) = p_4(T^*) = 0 \tag{3.2}$$

yields

$$T^* = \frac{\pi_3}{\pi_1} = \frac{\pi_4}{\pi_2} \tag{3.3}$$

or

$$\pi_1\pi_4 - \pi_2\pi_3 = 0 \tag{3.4}$$

Now consider the relation (2.13)

$$\tan \psi^*(t) = \frac{p_4(t)}{p_3(t)} \tag{3.5}$$

By taking time-derivatives of both sides of Eq. (3.5) and by using Eq. (2.3) and (3.1) one obtains

$$\frac{d}{dt} \tan \psi^*(t) = \frac{-\dot{p}_3(t)p_4(t) + p_3(t)\dot{p}_4(t)}{p_3^2(t)}$$

$$= \frac{p_1(t)p_4(t) - p_2(t)p_3(t)}{p_3^2(t)} = \frac{\pi_1(\pi_4 - \pi_2 t) - \pi_2(\pi_3 - \pi_1 t)}{p_3^2(t)}$$

$$= \frac{\pi_1\pi_4 - \pi_2\pi_3}{p_3^2(t)} = 0 \tag{3.6}$$

in view of Eq. (3.4). This implies that

$$\tan \psi^*(t) = \text{constant} \tag{3.7}$$

and

$$\psi^*(t) = \psi^* = \text{constant} \tag{3.8}$$

The fact that ψ^* is constant represents useful information. It cannot be as yet evaluated because one must determine the thrust $f^*(t)$. For this reason, the properties of $f^*(t)$ will be determined. It will be shown below that

(a) the singular condition (2.17c) does not occur,

(b) the optimal thrust $f^*(t)$ is full-on ($f^*(t) = F$) at the beginning of the time-interval until all the fuel is consumed at which time it is turned off ($f^*(t) = 0$).

The fact that the singular condition (2.17c) cannot occur is proved by contradiction. Let $\hat{\tau}$ denote a subinterval of $[0,T^*]$. Suppose that the switching function $s(t)$ is identically zero for all $t \in \hat{\tau}$. Thus, suppose (see Eq. (2.15))

$$s(t) = \frac{p_3(t)}{m(t)} \cos \psi^* + \frac{p_4(t)}{m(t)} \sin \psi^* - \frac{p_5(t)}{c} = 0 \quad \text{for all } t \in \hat{\tau} \tag{3.9}$$

where we have used the fact that $\psi^* =$ constant. Since the Hamiltonian must be zero along the optimal trajectory (see Eq. (2.8)), substitution of (3.9) into (2.16) yields

$$1 + v_x(t)\, \pi_1 + v_y(t)\, \pi_2 = 0 \qquad \text{for all } t \in \hat{\tau} \tag{3.10}$$

Differentiation of (3.10) and substitution of (1.5) yields

$$\frac{f(t)}{m(t)}\, \pi_1 \cos \psi^* + \frac{f(t)}{m(t)}\, \pi_2 \sin \psi^* = 0 \qquad \text{for all } t \in \hat{\tau} \tag{3.11}$$

Hence, (3.11) implies

$$\tan \psi^* = -\frac{\pi_1}{\pi_2} \tag{3.12}$$

But

$$\tan \psi^* = \frac{p_4(t)}{p_3(t)} = \frac{\pi_4 - \pi_2 t}{\pi_3 - \pi_1 t} \tag{3.13}$$

Hence (3.12) and (3.13) imply that

$$\left[\pi_1^2 + \pi_2^2\right] t = \pi_1\pi_3 + \pi_2\pi_4 \tag{3.14}$$

which can hold <u>if and only if</u>

$$\pi_1 = \pi_2 = 0 \tag{3.15}$$

But substitution of (3.15) into (3.10) yields $0 = 1$, obviously a contradiction. This implies that the hypothesis $s(t) = 0$ for <u>all</u> $t \in \mathcal{T}$ is false; therefore, the singular condition (2.17c) cannot occur. This in turn implies that the optimal thrust $f^*(t)$ can only attain its full-on value $f^*(t) = F$ or its cutoff value $f^*(t) = 0$.

The next topic to be investigated deals with the "shape" of $f^*(t)$. Intuitively, one would expect that the thrust should be turned-on as soon as possible so as to attain the maximum velocity possible. This intuitive feeling can indeed be verified using the necessary conditions.

To deduce the switching nature of $f^*(t)$ one examines the switching function $s(t)$ (see Eq. (3.9)). Differentiation of $s(t)$ and use of $\dot{m}(t) = -\frac{f^*(t)}{c}$ and of (2.3) yields

$$\begin{aligned}\dot{s}(t) = \frac{1}{m^2(t)} \Bigg[&\frac{f^*(t)}{c} [p_3(t) \cos \psi^* + p_4(t) \sin \psi^*] \\ &+ m(t) [\dot{p}_3(t) \cos \psi^* + \dot{p}_4(t) \sin \psi^*] \Bigg] \\ &- \frac{f^*(t)}{m^2(t)c} [p_3(t) \cos \psi^* + p_4(t) \sin \psi^*]\end{aligned} \tag{3.16}$$

and so, Eq. (3.16) reduces to

$$\dot{s}(t) = \frac{1}{m(t)} [\pi_1 \cos \psi^* + \pi_2 \sin \psi^*] \tag{3.17}$$

We compute $\ddot{s}(t)$ to deduce that

$$\ddot{s}(t) = \frac{f^*(t)}{m^2(t)c} [\pi_1 \cos \psi^* + \pi_2 \sin \psi^*] \tag{3.18}$$

From Eqs. (3.17) and (3.18) we obtain the relation

$$\ddot{s}(t) = \frac{f^*(t)}{m(t)c} \dot{s}(t) \tag{3.19}$$

which is integrated to yield

$$\dot{s}(t) = \dot{s}(\tau) \exp\left[\int_\tau^t \frac{f^*(\sigma)}{m(\sigma)c} d\sigma\right] \tag{3.20}$$

Since the exponential function is always positive, it follows that $\dot{s}(t)$ always has the same sign (positive or negative) for all $t \in [0,T^*]$.

At the terminal time T^*, the switching function $s(T^*)$ is given by

$$s(T^*) = \frac{1}{m(T^*)} [p_3(T^*) \cos \psi^* + p_4(T^*) \sin \psi^*] - \frac{p_5(T^*)}{c} \tag{3.21}$$

Since $p_3(T^*) = p_4(T^*) = 0$ (see Eq. (2.6)) it follows that

$$s(T^*) = - \frac{p_5(T^*)}{c} \tag{3.22}$$

We claim that

$$s(T^*) > 0 \tag{3.23}$$

To see this suppose the contrary, i.e., that $s(T^*) < 0$. Then, Eq. (3.22) yields $p_5(T^*) > 0$. But let us recall that

$$\dot{p}_5(t) = \frac{f^*(t)}{m^2(t)} [p_3(t) \cos \psi^* + p_4(t) \sin \psi^*] \leq 0 \tag{3.24}$$

in view of the necessary condition (2.14). Hence, if $p_5(T^*)$ 0, then Eq. (3.24) implies that

$$p_5(t) > 0 \qquad \text{for all } t \in [0,T^*] \tag{3.25}$$

Since the switching function s(t) is given by

$$s(t) = \frac{1}{m(t)} [p_3(t) \cos \psi^* + p_4 \sin \psi^*] - \frac{p_5(t)}{c} \tag{3.26}$$

then the necessary condition (2.14) and Eq. (3.25) yield

$$s(t) < 0 \qquad \text{for all } t \in [0,T^*] \tag{3.27}$$

But this means (see Eq. (2.17a)) that $f^*(t) = F$ for all $t \in [0,T^*]$, i.e., that full-thrust is applied throughout the control interval; this clearly violates the fuel constraint and Eq. (2.7). Therefore, $s(T^*) < 0$ cannot occur and, so, Eq. (3.23) must be true.

It was shown above that $s(T^*) > 0$. We also demonstrated that $\dot{s}(t)$ always has the same sign. If $\dot{s}(t) < 0$ for all $t \in [0,T^*]$, then $s(t) < 0$ for all $t \in [0,T^*]$, which means that $f^*(t) = 0$ for all $t \in [0,T^*]$; but if no thrust is applied, then interception in general is not possible. It, therefore follows that

$$\dot{s}(t) > 0 \qquad \text{for all } t \in [0,T^*] \tag{3.28}$$

and, hence, s(t) is a monotonically increasing time-function. This means that $s(t) < 0$ during the beginning of the time-interval $[0,T^*]$, crosses zero at some time $t = t_c$ and then remains positive. This establishes the fact that one must thrust in the beginning until all the fuel is exhausted (at $t = t_c$) and then coast, i.e.,

$$\left.\begin{array}{lll} f^*(t) = F & \text{for all } t \in [0,t_c] \\ f^*(t) = 0 & \text{for all } t \in (t_c,T^*] \end{array}\right\} \tag{3.29}$$

To recapitulate: It has been proven that the optimal thrusting program is to apply full thrust as soon as possible; during this thrusting period, the angle $\psi(t)$ of the thrust must remain constant. In the following section this angle will be found as a function of the state variables.

The thrust cutoff-time t_c can be computed because at $t = t_c$ all the available fuel has been exhausted. Since $f(t) = F$ for $t \in [0,t_c]$, the differential equation for the mass is

$$\dot{m}(t) = -\frac{f(t)}{c} = -\frac{F}{c} \tag{3.30}$$

At $t = 0$, $m(t_o) \triangleq m_o$ while at $t = t_c$, $m(t_c) \triangleq m_e$. Thus,

$$m(t_c) = m_e = m_o - \frac{F}{c} t_c \tag{3.31}$$

and, hence, the cutoff time t_c is given by

$$t_c = \frac{[m_o - m_e]\, c}{F} \tag{3.32}$$

The coasting period is $[t_c,T^*]$.

IV. A CLOSED-FORM EXPRESSION FOR THE THRUSTING ANGLE ψ^*

Let us examine the situation at the thrust cutoff time t_c. Let $e_x(t_c)$, $e_y(t_c)$, $v_x(t_c)$, $v_y(t_c)$ and $m(t_c) = m_e$ denote the values of the five state variables at t_c. In order to accomplish interception at $t = T^*$, one must have

$$e_x(T^*) = e_y(T^*) = 0 \tag{4.1}$$

Since there is no thrusting for $t \in [t_c, T^*]$, it follows that the velocity vector at $t_c \begin{bmatrix} v_x(t_c) \\ v_y(t_c) \end{bmatrix}$ must point toward the origin of the $e_x - e_y$ plane (see Fig. 2). This implies that at $t = t_c$ the following relations must hold

$$e_x(t_c)\ v_y(t_c) = e_y(t_c)\ v_x(t_c) \tag{4.2}$$

$$\left.\begin{aligned} e_x(t_c)\ v_x(t_c) < 0 \\ e_y(t_c)\ v_y(t_c) < 0 \end{aligned}\right\} \tag{4.3}$$

These relations together with $m(t_c) = m_e$ define a surface S_c in the five-dimensional state space. Clearly, if the state belongs to S_c, then interception is possible. In fact, the time $T^* - t_c$ can be evaluated analytically and is given by

$$T^* - t_c = - \frac{e_x(t_c)}{v_x(t_c)} = - \frac{e_y(t_c)}{v_y(t_c)} \tag{4.4}$$

It should be now clear how one can determine the thrust angle ψ^* prior to the cutoff time t_c. Let $t \in [0, t_c]$ and let $e_x(t)$, $e_y(t)$, $v_x(t)$, $v_y(t)$, and $m(t)$ be the values of the five state variables at that time. Since F and ψ^* are constant from t to t_c, the differential equations (1.5) are

$$\left.\begin{aligned} \dot{e}_x(t) &= v_x(t) \\ \dot{e}_y(t) &= v_y(t) \\ \dot{v}_x(t) &= \frac{F}{m(t)} \cos \psi^* \\ \dot{v}_y(t) &= \frac{F}{m(t)} \sin \psi^* \\ \dot{m}(t) &= - \frac{F}{c} \end{aligned}\right\} \tag{4.5}$$

These equations can be integrated from t to t_c to yield:

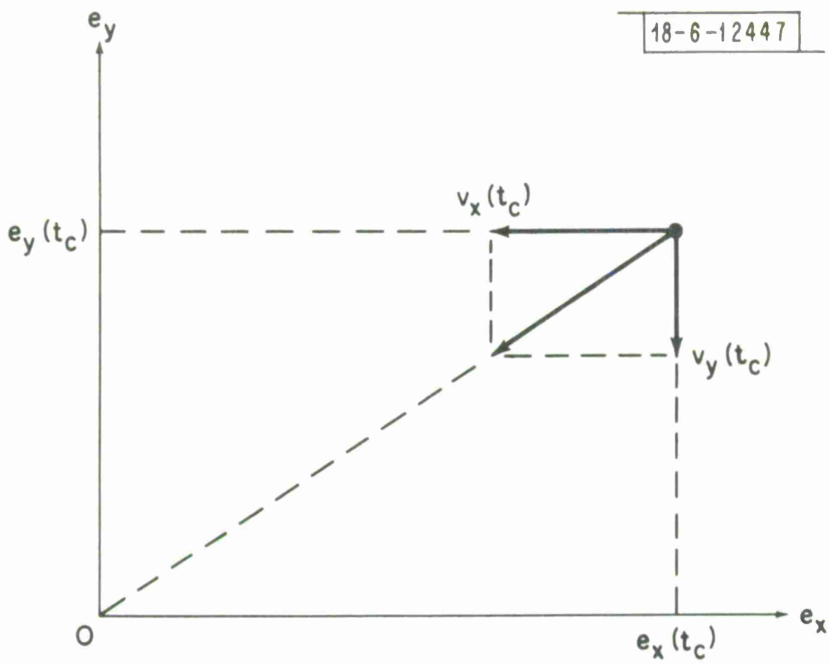

Fig. 2. Relation of velocities and errors at cutoff to guarantee interception.

$$m(t_c) \stackrel{\triangle}{=} m_e = m(t) - \frac{F}{c}(t_c - t) \tag{4.6}$$

$$v_x(t_c) = v_x(t) + \int_t^{t_c} \frac{F \cos \psi^*}{m(\tau)} d\tau = v_x(t) + F \cos \psi^* \cdot \int_t^{t_c} \frac{1}{m(t) - \frac{F}{c}(\tau - t)} d\tau$$

$$= v_x(t) + F \cos \psi^* \cdot \frac{c}{F} \log \left[\frac{m(t)}{m(t) - \frac{F}{c}(t_c - t)}\right] \tag{4.7}$$

Using (4.6) one finds that

$$v_x(t_c) = v_x(t) + c \log \left(\frac{m(t)}{m_e}\right) \cdot \cos \psi^* \tag{4.8}$$

In an identical manner one obtains

$$v_y(t_c) = v_y(t) + c \log \left(\frac{m(t)}{m_e}\right) \cdot \sin \psi^* \tag{4.9}$$

Finally

$$e_x(t_c) = e_x(t) + \int_t^{t_c} v_x(\tau)\, d\tau = e_x(t) + \int_t^{t_c} \left\{ v_x(t) + c \log \left[\frac{m(t)}{m(t) - \frac{F}{c}(\tau - t)}\right] \cos \psi^* \right\} d\tau$$

$$= e_x(t) + v_x(t)[t_c - t] + \cos \psi^* . c \int_t^{t_c} \log \left[\frac{m(t)}{m(t) - \frac{F}{c}(\tau - t)}\right] d\tau \tag{4.10}$$

Evaluation of the integral and use of (4.6) yields

$$e_x(t_c) = e_x(t) + v_x(t)[t_c - t] + c[t_c - t] \log m(t) \cos \psi^* + c[t_c - t] \cos \psi^*$$
$$+ \frac{c^2}{F} m_e \log m_e \cdot \cos \psi^* - \frac{c^2}{F} m(t) \log m(t) \cdot \cos \psi^* \tag{4.11}$$

But from (4.6)

$$t_c - t = \frac{c}{F}[m(t) - m_e] \tag{4.12}$$

Therefore, $e_x(t_c)$ can be simplified to

$$e_x(t_c) = e_x(t) + \frac{c}{F}[m(t) - m_e]\, v_x(t) + \frac{c^2}{F}\left[m_e \log \frac{m_e}{m(t)} + m(t) - m_e\right]\cos \psi^* \tag{4.13}$$

In a similar manner one obtains

$$e_y(t_c) = e_y(t) + \frac{c}{F}[m(t) - m_e]\, v_y(t) + \frac{c^2}{F}\left[m_e \log \frac{m_e}{m(t)} + m(t) - m_e\right]\sin \psi^* \tag{4.14}$$

Substituting Eqs. (4.14), (4.13), (4.8), and (4.9) into Eq. (4.2) one obtains the equation

$$\begin{aligned}\left[e_x(t) + \frac{c}{F}[m(t) - m_e]\, v_x(t) + \frac{c^2}{F}\, q(t) \cos \psi^*\right]\cdot\left[v_y(t) + c \log\left[\frac{m(t)}{m_e}\right]\cdot \sin \psi^*\right] \\ = \left[e_y(t) + \frac{c}{F}[m(t) - m_e]\, v_y(t) + \frac{c^2}{F}\, q(t) \sin \psi^*\right]\cdot\left[v_x(t) + c \log\left[\frac{m(t)}{m_e}\right]\cdot \cos \psi^*\right]\end{aligned} \tag{4.15}$$

where $q(t)$ is defined for the sake of notational simplicity by

$$q(t) \triangleq - m_e \log\left(\frac{m(t)}{m_e}\right) + m(t) - m_e \tag{4.16}$$

By performing the indicated multiplications, Eq. (4.15) reduces to

$$\begin{aligned}\left[\delta(t)e_x(t) + \gamma(t)v_x(t)\right] \sin \psi^* - \left[\delta(t)e_y(t) + \gamma(t)v_y(t)\right]\cos \psi^* \\ = e_y(t)v_x(t) - e_x(t)v_y(t)\end{aligned} \tag{4.17}$$

where $\delta(t)$ and $\gamma(t)$ are defined by

$$\delta(t) \triangleq c \log \left(\frac{m(t)}{m_e} \right) \tag{4.18}$$

$$\gamma(t) \triangleq \frac{c^2}{F} \left[m(t) \log \left(\frac{m(t)}{m_e} \right) - m(t) + m_e \right] \tag{4.19}$$

In this manner, the problem reduces to the solution of Eq. (4.17) for ψ^*. We note that Eq. (4.17) is of the form

$$a_1(t) \sin \psi^* + a_2(t) \cos \psi^* = b(t) \tag{4.20}$$

where

$$a_1(t) \triangleq \delta(t) e_x(t) + \gamma(t) v_x(t) \tag{4.21}$$

$$a_2(t) \triangleq - \left[\delta(t) e_y(t) + \gamma(t) v_y(t) \right] \tag{4.22}$$

$$b(t) \triangleq e_y(t) v_x(t) - e_x(t) v_y(t) \tag{4.23}$$

But

$$a_1(t) \sin \psi^* + a_2(t) \cos \psi^* = \sqrt{a_1^2(t) + a_2^2(t)} \sin \left[\psi^* + \tan^{-1} \frac{a_2(t)}{a_1(t)} \right] \tag{2.24}$$

Therefore, from Eqs. (4.24) and (4.20) we deduce the desired solution

$$\psi^* = \sin^{-1} \frac{b(t)}{\sqrt{a_1^2(t) + a_2^2(t)}} - \tan^{-1} \frac{a_2(t)}{a_1(t)} \tag{4.25}$$

If the relation

$$\left| \frac{b(t)}{\sqrt{a_1^2(t) + a_2^2(t)}} \right| \leq 1 \tag{4.26}$$

holds, then interception is possible. Otherwise, the interceptor will miss the target.

The complete structure of the control system that generates the optimal thrust angle ψ^* is illustrated in block diagram form in Figs. 3 through 6 .

References

1. M. Athans and P.L. Falb, Optimal Control, McGraw Hill Book Company, New York, New York, 1966.

2. G. Leitman (Editor), Optimization Techniques, Academic Press, New York, 1962.

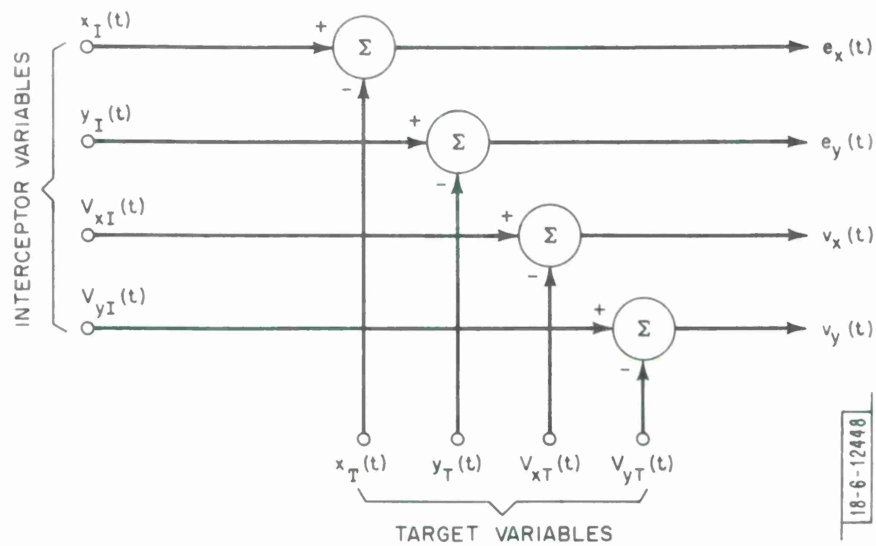

Fig. 3. Generation of error variables [cf. Eq. (1.4)].

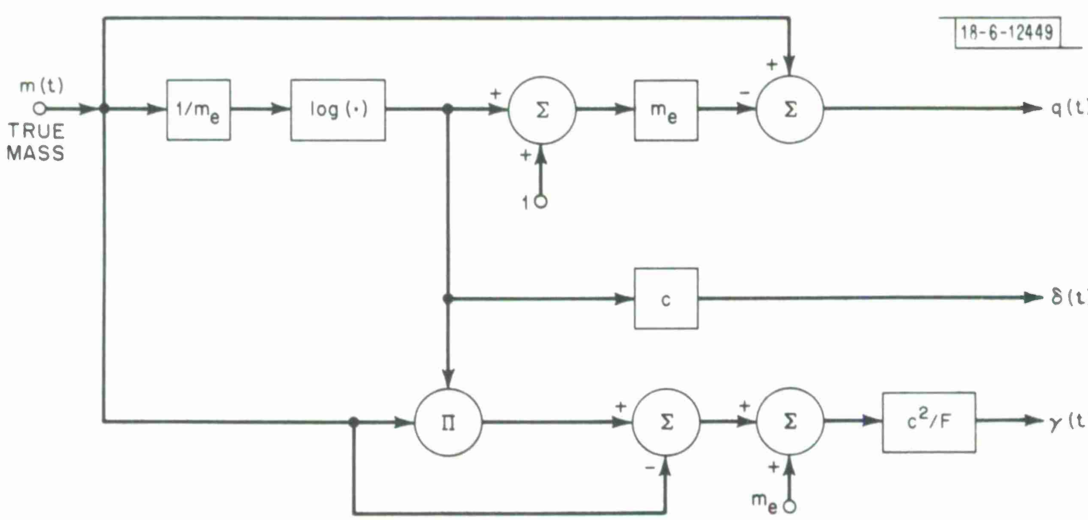

Fig. 4. Generation of auxiliary time variables q(t), δ(t), and γ(t) [cf. Eqs. (4.16), (4.18), (4.19)]. m_e is the empty interceptor mass, c is the equivalent exit velocity, and F is the maximum thrust.

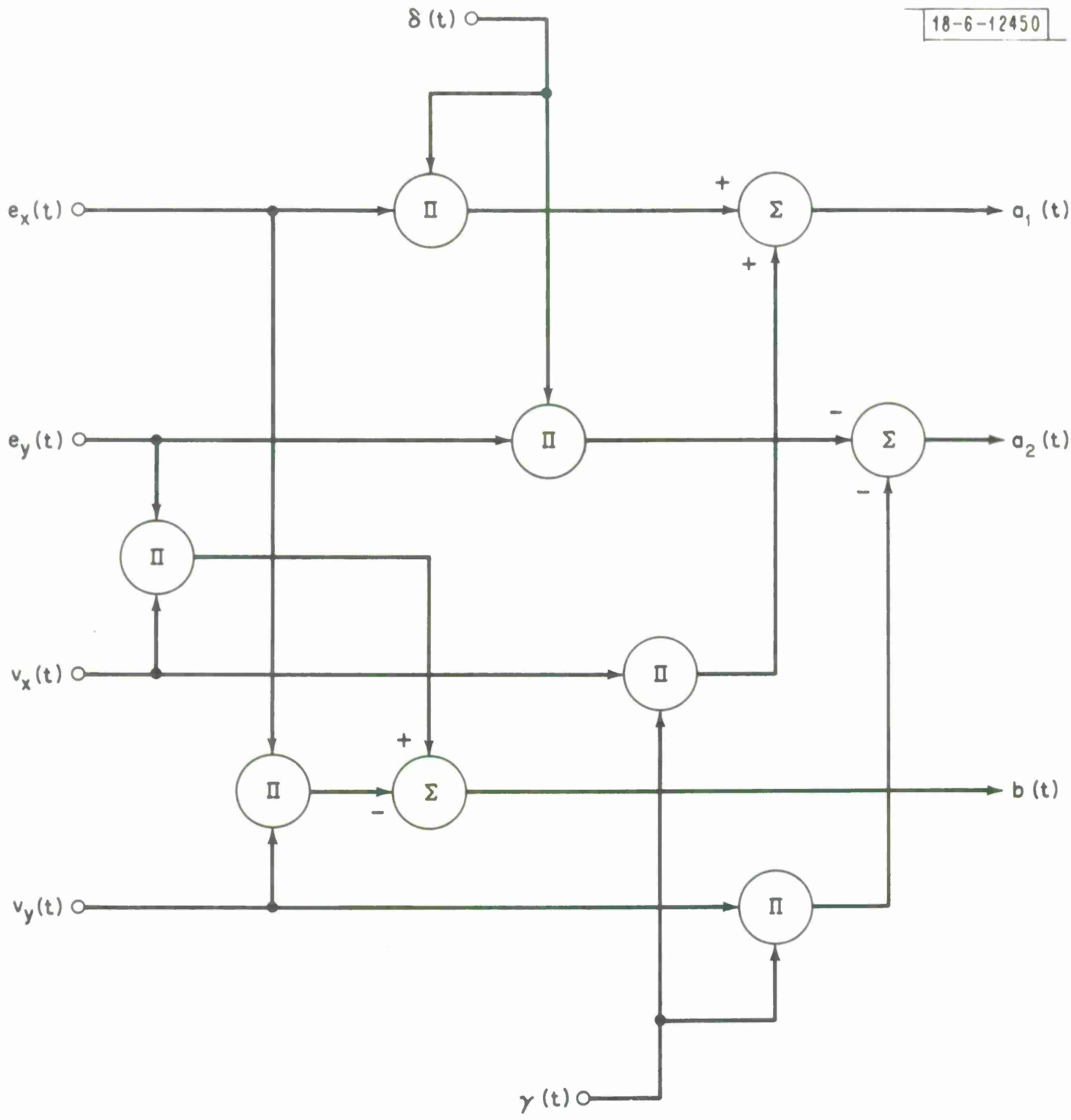

Fig. 5. Generation of auxiliary time functions $a_1(t)$, $a_2(t)$, and $b(t)$ [cf. Eqs. (4.21), (4.22), (4.23)] from the error variables (see Fig. 3) and the auxiliary variables $\gamma(t)$ and $\delta(t)$ (see Fig. 4).

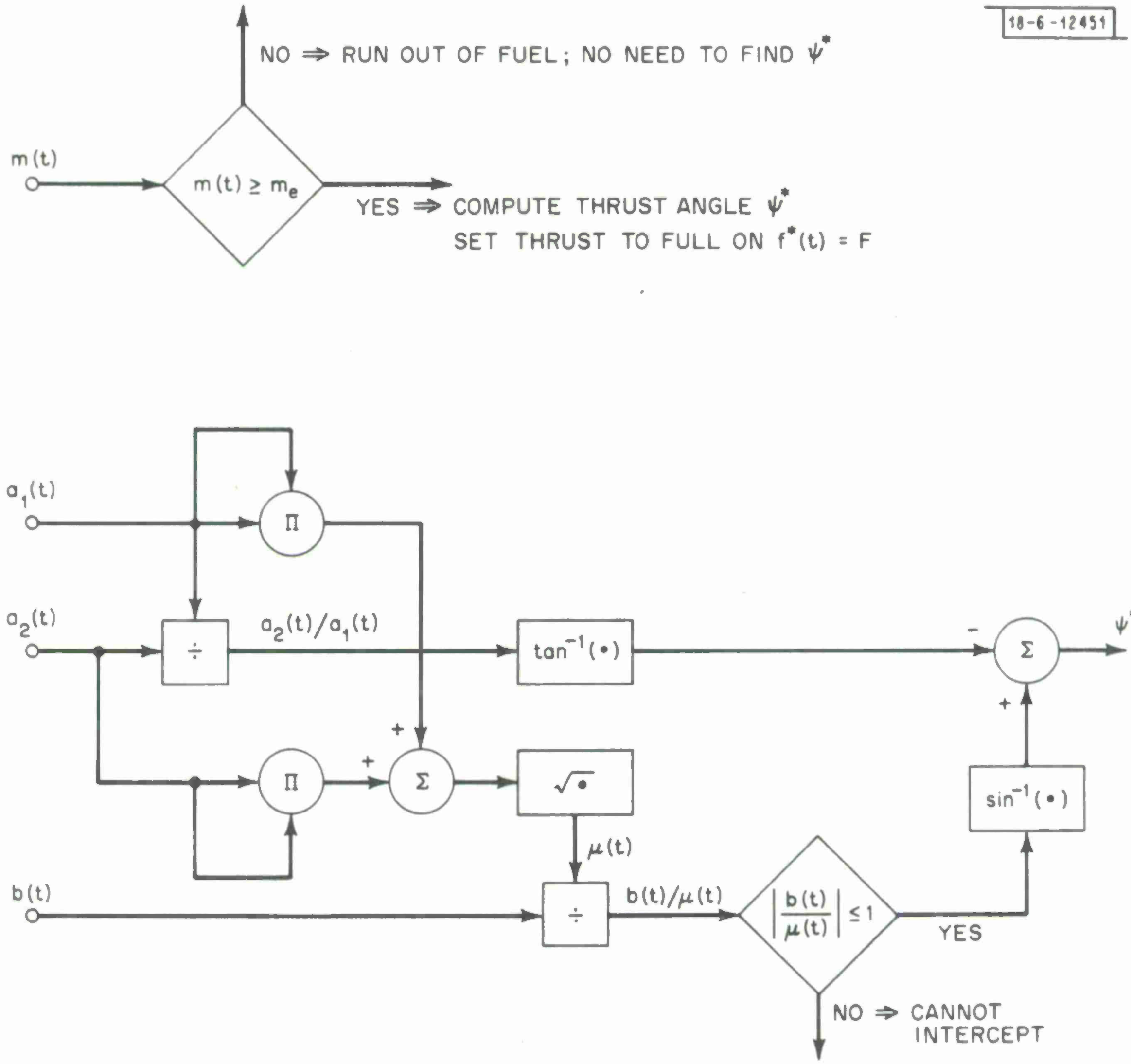

Fig. 6. Generation of optimal thrust and thrust direction angle ψ^*.

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13. ABSTRACT

The problem considered is that of minimum time interception, subject to fuel constraints, of a nonmaneuvering target by an interceptor. The motion of the interceptor is controlled by application of a thrust vector (with bounded magnitude) and by the control of the thrusting angle relative to, say, the interceptor velocity vector. This problem is solved using the minimum principle of Pontryagin[2]. It is shown that the time-optimal interception policy is (a) to use maximum thrust in the beginning of the control interval and (b) to use a constant thrust angle during the thrusting interval. The optimal thrust angle is determined analytically as a function of the position error, the velocities, and of the remaining fuel.

14. KEY WORDS

exoatmospheric interception
minimum time interception
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